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1978 J. Phys. A: Math. Gen. 11 1087

(http://iopscience.iop.org/0305-4470/11/6/011)

J. Phys. A: Math. Gen., Vol. 11, No. 6, 1978. Printed in Great Britain

The distorted wave Glauber approximation

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Received 15 September 1977, in final form 30 January 1978

Abstract. A solution of the Pauli equation with non-zero potentials defines quantum scalar and vector potentials and magnetic fields and quantum trajectories. If a line integral of perturbing potentials and fields along these quantum trajectories is added to the phase of this solution, an approximate solution of the perturbed equation is found. Glauber theory is a special case and the conditions of applicability are similar. Applications given start from the harmonic oscillator and from a homogeneous magnetic field and add a perturbation.

1. Introduction

The quantum forces defined in § 2 are as old as quantum mechanics (de Broglie 1930) and persist to the present day in attempts to provide a mechanistic or hydrodynamical explanation of quantum mechanics. Section 5 which introduces quantum vector potentials in such a treatment of the Pauli equation may be regarded as a contribution to this study.

Our interest in the quantum potential $V_q(\boldsymbol{x})$ is, however, in retaining the quantum or interference aspects of the known solution $a(\boldsymbol{x})\,e^{iS(\boldsymbol{x})/\hbar}$ of the Schrödinger equation with potential $V(\boldsymbol{x})$. The Schrödinger equation is entirely equivalent to a coupled pair of equations: the Hamilton–Jacobi (HJ) equation with a quantum potential $V_q(\boldsymbol{x})$ in addition to $V(\boldsymbol{x})$; and the continuity equation. The HJ equation can be regarded as determining the phase $S(\boldsymbol{x})$ and the continuity equation as determining the amplitude $a(\boldsymbol{x})$ or the current density $a^2(\boldsymbol{x})\nabla S(\boldsymbol{x})/m$. But they are coupled because the quantum potential

$$V_q(\boldsymbol{x})=\frac{\hbar^2}{2m}\frac{1}{a(\boldsymbol{x})}\nabla^2 a(\boldsymbol{x}) \tag{1.1}$$

depends on $a(\boldsymbol{x})$ and the current depends on $S(\boldsymbol{x})$.

It may happen that the potential $\tilde{V}(\boldsymbol{x})$ in the Schrödinger equation is too large for the Glauber approximation to be used, but that we can exactly solve the scattering problem for the potential $V(\boldsymbol{x})$ which differs from $\tilde{V}(\boldsymbol{x})$ in the neighbourhood of $\boldsymbol{x}=0$ but not asymptotically.

The transition amplitude

$$\tilde{T}_{\boldsymbol{k}\to\boldsymbol{k}'}=\langle e^{-i\boldsymbol{k}'.\boldsymbol{x}}|\tilde{V}(\boldsymbol{x})|\tilde{\psi}^+_{\boldsymbol{k}}\rangle \tag{1.2}$$

where

$$\tilde{H}\tilde{\psi}^+_{\boldsymbol{k}}=E\tilde{\psi}^+_{\boldsymbol{k}} \tag{1.3}$$

$$\tilde{\psi}^+_{\boldsymbol{k}}\underset{|\boldsymbol{x}|\to\infty}{\longrightarrow} e^{i\boldsymbol{k}.\boldsymbol{x}}+\tilde{f}_{\boldsymbol{k}',\boldsymbol{k}}\,e^{ikr}/r, \qquad |\boldsymbol{k}|=|\boldsymbol{k}'|=k;\quad |\boldsymbol{x}|=r \tag{1.4}$$

0305-4770/78/0006-1087$03.00

is given in terms of the transition amplitude $T_{k\to k'}$ for $V(x)$, by

$$\tilde{T}_{k\to k'} = T_{k\to k'} - \langle \psi_{k'}^{-} | \tilde{V}(x) - V(x) | \tilde{\psi}_{k}^{+} \rangle \tag{1.5}$$

where

$$H\psi_{k'}^{-} = E\psi_{k'}^{-} \tag{1.6}$$

$$\psi_{k'}^{-} = a^{-}(x)\, e^{iS^{-}(x)/\hbar} \xrightarrow[r\to\infty]{} e^{ik'.x} + f_{k,k'}\, e^{-ikr}/r \tag{1.7}$$

In § 2 we shall approximate $\tilde{\psi}_{k}^{+}(x)$ the only unknown in (1.5) by $\psi_{k}^{+}(x)$ modulated by a factor $\exp[(i/\hbar)(\int^{x} m(\tilde{V}-V)(\nabla S.\nabla S)^{-1/2}\, dl)]$ where the line integral is taken along the trajectory tangent to $\nabla S(x)$, of a particle which experiences a 'quantum force' in addition to $-\nabla V(x)$. The approximation is valid provided $\tilde{V}(x)-V(x)$ is a smooth function. Since this function is zero asymptotically the smoothness condition restricts the magnitude of $\tilde{V}(x)-V(x)$ compared with the kinetic energy $\nabla S.\nabla S/2m$. The modulating factor is unity, to the asymptotic side of some surface, the crest of an incoming wave say, so that corrections to the backward scattering have been neglected. This is a high energy forward scattering approximation and reduces to Glauber's (1958) when $V(x)$ is zero (then the trajectories are straight lines). It differs from the classical approximation (WKB) in that (i) it includes, by means of the quantum potential for $V(x)$ and $\psi_{k}^{+}(x)$, the interference effects appropriate to $V(x)$ and (ii) the line integrals are not along the classical trajectories but along quantum trajectories which take the 'quantum force' into account. It differs from the modifications of Glauber theory which replace his straight line trajectories, along which $\tilde{V}$ is integrated, by the classical trajectories for $\tilde{V}(x)$. We suggest that these modifications have been less successful than pure Glauber theory because Glauber's straight trajectories are quantum trajectories for the case $V=0$ (because $a(x)$ is constant and $V_q(x)$ zero) whereas the classical trajectories lack information on the quantum aspects of the unperturbed as well as the perturbed system.

Absorptive parts to the potentials are introduced in § 3 leading to slight modifications of the results of § 2. The application to separable potentials is discussed.

The time dependent problem is tackled in § 4 in the presence of vector and scalar potentials and applied to perturbations of an harmonic oscillator. In § 5 quantum vector potentials and independent magnetic fields are introduced in order to accommodate spin and in § 6 the method is used to treat the time independent spin case; the time dependent case is in the appendix. The scattering of a charged particle with spin undergoing a nuclear force with spin–orbit coupling and exchange in the presence of a strong homogeneous magnetic field is treated in § 7. This problem is of interest in neutron stars.

2. The Schrödinger equation

The Schrödinger equation

$$-H\psi(x) \equiv (\tfrac{1}{2}\hbar^2 m^{-1}\nabla^2 - V(x))\psi(x) = -E\psi(x) \tag{2.1}$$

is satisfied by

$$\psi(x) = a(x)\exp(i\hbar^{-1}S(x)) \tag{2.2}$$

with real functions $a(\boldsymbol{x})$, $S(\boldsymbol{x})$ if and only if

$$\nabla S \cdot \nabla S + 2mV - \hbar^2 a^{-1} \nabla^2 a = 2Em \tag{2.3}$$

and

$$\nabla \cdot (a^2 \nabla S) = 0 \tag{2.4}$$

simultaneously. Equation (2.3) may be interpreted as the Hamilton–Jacobi (HJ) equation for the 'momentum' $\nabla S(\boldsymbol{x})$ at $\boldsymbol{x}$ in the presence of a quantum force $\frac{1}{2}\hbar^2 m^{-1} \nabla a^{-1} \nabla^2 a$ in addition to the classical force $-\nabla V$ and (2.4) as the continuity equation for the density $a^2(\boldsymbol{x})$ of particles moving with velocity field $m^{-1}\nabla S(\boldsymbol{x})$ in this force field.

For the Schrödinger equation with potential $\tilde{V}(\boldsymbol{x})$ the equation will have a different classical and quantum potential. We look for that solution which is equal to $\psi(\boldsymbol{x})$ in the incoming asymptotic region, say

$$\tilde{\psi}(\boldsymbol{x}) = \psi(\boldsymbol{x}) \qquad \text{along } S(\boldsymbol{x}) = c \tag{2.5}$$

$$\nabla \tilde{\psi}(\boldsymbol{x}) = \nabla \psi(\boldsymbol{x}) \qquad \text{along } S(\boldsymbol{x}) = c. \tag{2.6}$$

Writing

$$\tilde{\psi}(\boldsymbol{x}) = (a(\boldsymbol{x}) + a_1(\boldsymbol{x})) \exp(\mathrm{i}\hbar^{-1} \tilde{S}(\boldsymbol{x})) \tag{2.7}$$

the HJ equation is

$$\nabla \tilde{S} \cdot \nabla \tilde{S} + 2m(\tilde{V} - E) + \hbar^2 (a + a_1)^{-1} \nabla^2 (a + a_1) = 0 \tag{2.8}$$

and the continuity equation

$$\nabla \cdot [(a + a_1)^2 \nabla \tilde{S}] = 0. \tag{2.9}$$

We can decouple the HJ and continuity equations if in the HJ equation we make the *ansatz I* of neglecting a_1 and in the continuity equation we make the *ansatz II* of insisting that surfaces of constant $S(\boldsymbol{x})$ are also surfaces of constant $\tilde{S}(\boldsymbol{x})$:

$$\nabla \tilde{S}(\boldsymbol{x}) = f(S(\boldsymbol{x})) \nabla S(\boldsymbol{x}). \tag{2.10}$$

These *ansätze* are consistent with the asymptotic conditions (2.5) and (2.6) if f is one asymptotically.

We shall now show that *ansätze* I and II may both be used consistently in both equations (2.8) and (2.9), provided f is given by

$$1 - f^2(S(\boldsymbol{x})) = 2m(\tilde{V}(\boldsymbol{x}) - V(\boldsymbol{x}))/(\nabla S(\boldsymbol{x}))^2 \tag{2.11}$$

and its gradient along ∇S is negligible. This in turn puts a restriction on the type of potential $\tilde{V}(\boldsymbol{x})$ which may be treated by the method.

Imposing both *ansätze*

$$\tilde{\psi}(\boldsymbol{x}) = a(\boldsymbol{x}) \exp(\mathrm{i}\hbar^{-1} \tilde{S}(\boldsymbol{x})) \tag{2.12}$$

where $\tilde{S}(\boldsymbol{x})$ satisfies (2.10). The HJ equation

$$\nabla \tilde{S} \cdot \nabla \tilde{S} + 2m(\tilde{V} - E) - \hbar^2 a^{-1} \nabla^2 a = 0 \tag{2.13}$$

will be satisfied by virtue of (2.3) and (2.10) if f is given by (2.11). Using (2.10) the HJ equation has the solution

$$\tilde{S}(\boldsymbol{x}) - c = (2m)^{1/2} \int^{\boldsymbol{x}} (E - \tilde{V} + \tfrac{1}{2}\hbar^2 m^{-1} a^{-1} \nabla^2 a)^{1/2} \, \mathrm{d}l \tag{2.14}$$

where the reduced action integral is taken along the path tangent to $\nabla S(\boldsymbol{x})$, from $\boldsymbol{x}$ to the point where this path crosses $S(\boldsymbol{x})=c$, and $\mathrm{d}l$ is the increment of length along the path.

Using (2.4) and (2.10)

$$\nabla\cdot(a^2\nabla\tilde{S})=a^2\nabla S\cdot\nabla f \tag{2.15}$$

so that if the gradient of (2.11) along $\nabla S(\boldsymbol{x})$ is negligible the continuity equation

$$\nabla\cdot(a^2\nabla\tilde{S})=0 \tag{2.16}$$

is satisfied. This taken together with (2.13) implies that $\tilde{\psi}$ of (2.12) satisfies the Schrödinger equation

$$-\tilde{H}\tilde{\psi}(\boldsymbol{x})\equiv(\tfrac{1}{2}\hbar^2m^{-1}\nabla^2-\tilde{V}(\boldsymbol{x}))\tilde{\psi}(\boldsymbol{x})=-E\tilde{\psi}(\boldsymbol{x}). \tag{2.17}$$

The conditions that $\tilde{V}(\boldsymbol{x})$ must satisfy in order that the *ansätze* I and II may be valid, i.e. that $\tilde{\psi}$ given by (2.12) and (2.10) may satisfy the Schrödinger equation (2.17) with the asymptotic behaviour (2.5), (2.6), are as follows. $\tilde{V}(\boldsymbol{x})$ must not differ asymptotically from $V(\boldsymbol{x})$, in order that f may be one asymptotically and so (2.10) may be consistent with (2.5) and (2.6).

The component of the force $-\nabla\tilde{V}(\boldsymbol{x})$ perpendicular to $\nabla S(\boldsymbol{x})$ must be approximately $-f^2(S(\boldsymbol{x}))\nabla_\perp V(\boldsymbol{x})$ in order to balance the increased centrifugal force, if assumption (2.10) is to hold good:

$$\nabla_\perp\tilde{V}(\boldsymbol{x})-f^2\nabla_\perp V(\boldsymbol{x})\ll\nabla S(\boldsymbol{x})\cdot\nabla S(\boldsymbol{x})/2mL \tag{2.18}$$

where L is the typical length of a path tangent to $\nabla S(\boldsymbol{x})$ through the region in which $\tilde{V}(\boldsymbol{x})$ and $V(\boldsymbol{x})$ differ. A second smoothness condition on the potentials, since the gradient of $f(\boldsymbol{x})$ along $S(\boldsymbol{x})$ must be negligible, is

$$L\nabla S(\boldsymbol{x})\cdot\nabla 2m(\tilde{V}(\boldsymbol{x})-V(\boldsymbol{x}))/(\nabla S(\boldsymbol{x}))^2\ll|\nabla S(\boldsymbol{x})| \tag{2.19}$$

in order that (2.16) be approximately satisfied. Since $\tilde{V}(\boldsymbol{x})-V(\boldsymbol{x})$ is asymptotically zero this smoothness condition prevents $1-f^2$ from becoming large anywhere:

$$1-f^2(S(\boldsymbol{x}))=2m(\tilde{V}(\boldsymbol{x})-V(\boldsymbol{x}))/(\nabla S(\boldsymbol{x}))^2\ll 1. \tag{2.20}$$

We shall now work to first order in $1-f(S(\boldsymbol{x}))$, which, for L sufficiently large, allows us consistently to work to zero order in the gradients in (2.18) and (2.19), so that the continuity equation (2.16), and the solution (2.14) of the HJ equation, are valid.

To first order in $1-f$ this solution (2.14) becomes

$$\tilde{S}(\boldsymbol{x})=S(\boldsymbol{x})-\int^{\boldsymbol{x}}m(\tilde{V}(\boldsymbol{x})-V(\boldsymbol{x}))|\nabla S(\boldsymbol{x})|^{-1}\,\mathrm{d}l \tag{2.21}$$

where we have used the HJ equation (2.3) and its exact solution

$$S(\boldsymbol{x})-c=(2m)^{1/2}\int^{\boldsymbol{x}}(E-V+\tfrac{1}{2}\hbar^2m^{-1}a^{-1}\nabla^2a)^{1/2}\,\mathrm{d}l \tag{2.22}$$

called the abbreviated action or the integral of Maupertuis.

The integration in (2.21) and (2.22) is taken over the quantum trajectory, that is the path derived from the HJ equation (2.3) which includes a quantum potential. If this quantum potential were neglected (2.2) would be the WKB approximation.

The Glauber approximation is the particular case where $V(\boldsymbol{x})=0$, so that the trajectories are straight lines, parallel to the z axis say, then $S(\boldsymbol{x})$ is $\hbar kz$, $a(\boldsymbol{x})$ is unity and

$$\tilde{S}(\boldsymbol{b}, z)=S(\boldsymbol{b}, z)-m\int^{z} \mathrm{d}z'\,\tilde{V}(\boldsymbol{b}+\boldsymbol{n}z')/\hbar k \tag{2.23}$$

where $\boldsymbol{n}$ is a unit vector in the z direction and $\boldsymbol{b}$ is the impact parameter and lies in the x–y plane.

3. Absorptive and separable potentials

It is evident from our analysis in § 2 that $\tilde{V}$ may be allowed to be complex. If we add an imaginary potential $-\mathrm{i}V_a$ to V however it appears as an absorption term proportional to the density a^2 on the right of (2.4)

$$\hbar\nabla\cdot(a^2\nabla S)=-2ma^2V_a. \tag{3.1}$$

Thus

$$\hbar\nabla\cdot(a^2\nabla\tilde{S})=-2ma^2fV_a. \tag{3.2}$$

So the modified HJ equation

$$\tfrac{1}{2}m^{-1}\nabla\tilde{S}\cdot\nabla\tilde{S}+\tilde{V}-(1-f)\mathrm{i}V_a-E-\tfrac{1}{2}\hbar^2m^{-1}a^{-1}\nabla^2a=0 \tag{3.3}$$

must hold in order that $\tilde{\psi}$ satisfy the Schrödinger equation (2.17). The equation for $1-f$ becomes

$$\tfrac{1}{2}m^{-1}(1-f^2)(\nabla S)^2+(1-f)\mathrm{i}V_a=\tilde{V}-V \tag{3.4}$$

or

$$1-f=(\tilde{V}-V)/[m^{-1}(\nabla S)^2+\mathrm{i}V_a] \tag{3.5}$$

to first order in $1-f$ and (2.21) becomes

$$\tilde{S}(\boldsymbol{x})=S(\boldsymbol{x})-\int^{\boldsymbol{x}} m(\tilde{V}-V-(1-f)\mathrm{i}V_a)|\nabla S|^{-1}\,\mathrm{d}l. \tag{3.6}$$

The method can now be used even when V is non-local, for example a separable potential; for a given E and a given incoming wave, the potential can be represented by a complex local potential. For example, the separable potential

$$\langle\boldsymbol{p}|V|\boldsymbol{q}\rangle=-\tfrac{1}{2}\lambda(p^2+\beta^2)^{-1}(q^2+\beta^2)^{-1}m^{-1} \tag{3.7}$$

for energy $\frac{1}{2}k^2m^{-1}\hbar^2$ has the solution given by Yamaguchi (1954)

$$\psi(\boldsymbol{x})\propto\exp(\mathrm{i}kz)+[\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})-\exp(-\beta r)]g_kr^{-1} \tag{3.8}$$

where

$$g_k^{-1}=-\mathrm{i}k-\beta+\tfrac{1}{2}(\beta^2+k^2)\beta^{-1}+(\beta^2+k^2)^2(2\pi^2\lambda)^{-1}. \tag{3.9}$$

The same $\psi(\boldsymbol{x})$ is a solution of the Schrödinger equation with the local potential $V(\boldsymbol{x})$ where

$$V(\boldsymbol{x})=g_km^{-1}r^{-1}\tfrac{1}{2}(\beta^2+k^2)\hbar^2\exp(-\beta r). \tag{3.10}$$

Numerical tests of the method in this case by Atkin (1977) encourage us to try the method in the three-particle problem starting from the solution with pure separable two-particle potentials and adding a perturbation which makes them realistic. This should be sensitive to other aspects of the potentials than those tested by the Faddeev equations containing separable terms and a remainder (Alt *et al* 1967, Riordan 1968). It solves the three-particle Schrödinger equation directly and the only increase in complexity is that the points at which the wavefunction is to be calculated span a nine-dimensional space. The quantum trajectories in the nine-dimensional space are still defined paths (one-dimensional).

4. Time dependence. Vector potentials

4.1. The coupled HJ and continuity equations

We shall suppose in this section that we know the solution

$$\psi(\boldsymbol{x}, t) = a(\boldsymbol{x}, t)\exp(\mathrm{i}S(\boldsymbol{x}, t)/\hbar) \tag{4.1}$$

(where $a(\boldsymbol{x}, t)$ and $S(\boldsymbol{x}, t)$ are real functions) of the Schrödinger equation

$$\mathrm{i}\hbar\,\partial\psi(\boldsymbol{x}, t) = [\tfrac{1}{2}m^{-1}(-\mathrm{i}\hbar\boldsymbol{\nabla} - e\boldsymbol{A}(\boldsymbol{x}, t))^2 + eV(\boldsymbol{x}, t)]\psi(\boldsymbol{x}, t) \tag{4.2}$$

(∂ is the partial derivative with respect to t). Then $a(\boldsymbol{x}, t)$ and $S(\boldsymbol{x}, t)$ satisfy the coupled equations

$$m\,\partial a^2(\boldsymbol{x}, t) + \boldsymbol{\nabla}\cdot[(\boldsymbol{\nabla}S(\boldsymbol{x}, t) - e\boldsymbol{A}(\boldsymbol{x}, t))a^2(\boldsymbol{x}, t)] = 0, \tag{4.3}$$

$$2m\,\partial S(\boldsymbol{x}, t) + (\boldsymbol{\nabla}S(\boldsymbol{x}, t) - e\boldsymbol{A}(\boldsymbol{x}, t))^2 + 2meV(\boldsymbol{x}, t) - \hbar^2 a^{-1}(\boldsymbol{x}, t)\boldsymbol{\nabla}^2 a(\boldsymbol{x}, t) = 0. \tag{4.4}$$

We shall interpret (4.4) as the HJ equation for a particle of charge e in a scalar potential $V(\boldsymbol{x}, t) - \hbar^2 a^{-1}(\boldsymbol{x}, t)\boldsymbol{\nabla}^2 a(\boldsymbol{x}, t)/me$, and a vector potential $\boldsymbol{A}(\boldsymbol{x}, t)$.

The particular solution, Hamilton's principal function $S(\boldsymbol{x}, t)$, defines a bundle of world lines $\boldsymbol{x}_i(t)$, one through each point in coordinate space at time t, and may be written in terms of the action along them:

$$S(\boldsymbol{x}, t) - c_i = \int_{t_i}^{t} [\tfrac{1}{2}m(\dot{x}_i(t))^2 - eV(\boldsymbol{x}, t) - \tfrac{1}{2}\hbar^2 m^{-1} a^{-1}(\boldsymbol{x}, t)\boldsymbol{\nabla}^2 a(\boldsymbol{x}, t) + e\boldsymbol{A}(\boldsymbol{x}, t)\cdot\dot{x}_i(t)]_{\boldsymbol{x}_i(t)}\,\mathrm{d}t \tag{4.5}$$

$$c_i = S(\boldsymbol{x}_i(t_i), t_i), \tag{4.6}$$

t_i is a time when $\boldsymbol{x}_i(t_i)$ is asymptotic and $\boldsymbol{x}_i(t)$ is a world line of a particle which moves under the influence of these potentials and contains the event $\boldsymbol{x}, t$. That is

$$\boldsymbol{x} = \boldsymbol{x}_i(t) \tag{4.7}$$

$$m\dot{\boldsymbol{x}}_i(t) = [\boldsymbol{\nabla}S(\boldsymbol{x}, t) - e\boldsymbol{A}(\boldsymbol{x}, t)]_{\boldsymbol{x}_i(t)}. \tag{4.8}$$

We note in passing that the quantum potential is zero if $a(\boldsymbol{x}, t)$ is a function of time only, as for the propagator when the scalar potential is quadratic and the vector potential zero. The exact solution in this case (Morette 1951, Pauli 1973, Feynman and Hibbs 1965) is

$$\psi(\boldsymbol{x}, t) = a(t)\exp(\mathrm{i}S_{\mathrm{cl}}(\boldsymbol{x}, t)/\hbar) \tag{4.9}$$

where $S_{cl}(\boldsymbol{x}, t)$ is calculated without a quantum potential and $a(t)$ depends on the coefficient of the quadratic term in normal coordinates. This is the case when the WKB approximation is exact and the Feynman sum over paths may be replaced by a contribution of the classical path multiplied by $a(t)$.

4.2. *The approximation*

In approximating the solution of the perturbed system with potentials $\tilde{V}(\boldsymbol{x}, t)$, $\tilde{A}(\boldsymbol{x}, t)$ we shall suppose that the new world lines are $\boldsymbol{x}_i(\theta_i(t))$ where

$$1-\dot{\theta}_i^2(t)=2me[(\tilde{V}(\boldsymbol{x}, t)-V(\boldsymbol{x}, \theta)(\boldsymbol{\nabla} S(\boldsymbol{x}, \theta))-e\boldsymbol{A}(\boldsymbol{x}, \theta))^{-2}]_{\boldsymbol{x}_i(\theta),\theta_i(t)}, \qquad \theta_i(t_0)=t_0 \tag{4.10}$$

that is, the same *trajectories* as in the unperturbed system, but traversed at a different speed, so that a particle reaches a point on the trajectory $\boldsymbol{x}_i(t)$ at time $\theta_i(t)$, whereas it reached the same point at time t in the unperturbed system

$$\boldsymbol{\nabla}\tilde{S}(\boldsymbol{x}, t)-e\tilde{\boldsymbol{A}}(\boldsymbol{x}, t)=m\boldsymbol{x}_i'\dot{\theta}_i=\dot{\theta}(\boldsymbol{x}, t)[\boldsymbol{\nabla} S(\boldsymbol{x}, \theta)-e\boldsymbol{A}(\boldsymbol{x}, \theta)]_{\theta(\boldsymbol{x},t)} \tag{4.11}$$

where the prime denotes differentiation with respect to θ and we have defined

$$\theta(\boldsymbol{x}, t)=\theta_i(t) \qquad \text{for } \boldsymbol{x}, t \text{ on } \boldsymbol{x}_i(t) \tag{4.12}$$

and

$$\dot{\theta}(\boldsymbol{x}, t)=\dot{\theta}_i(t) \qquad \text{for } \boldsymbol{x}, t \text{ on } \boldsymbol{x}_i(t) \tag{4.13}$$

the dot denoting differentiation with respect to t and the subscript $\theta(\boldsymbol{x}, t)$ indicates here and throughout this paper that the gradient is to be taken with θ considered an independent variable, which is then set equal to $\theta(\boldsymbol{x}, t)$.

Further suppose that the energy at a point on the trajectory is the same in the perturbed and unperturbed systems

$$\partial\tilde{S}(\boldsymbol{x}, t)=[\partial_\theta S(\boldsymbol{x}, \theta)]_{\theta(\boldsymbol{x},t)} \tag{4.14}$$

where ∂_θ is the partial derivative with respect to θ. Then (4.10), (4.11) and the HJ equation (4.4) imply

$$2m\,\partial\tilde{S}(\boldsymbol{x}, t)+(\boldsymbol{\nabla}\tilde{S}(\boldsymbol{x}, t)-e\tilde{\boldsymbol{A}}(\boldsymbol{x}, t))^2+2me\tilde{V}(\boldsymbol{x}, t)-\hbar^2[a^{-1}(\boldsymbol{x}, \theta)\boldsymbol{\nabla}^2 a(\boldsymbol{x}, \theta)]_{\theta(\boldsymbol{x},t)}=0. \tag{4.15}$$

The Hamilton principal function $\tilde{S}(\boldsymbol{x}, t)$ defined for $\boldsymbol{x}, t$ on the world line $\boldsymbol{x}_i(\theta_i(t))$ by

$$\tilde{S}(\boldsymbol{x}, t)-c_i=\int_{t_i}^{t}[\tfrac{1}{2}m(\dot{\boldsymbol{x}}_i(\theta_i(t)))^2-eV(\boldsymbol{x}, t)-\tfrac{1}{2}\hbar^2 m^{-1}(a^{-1}(\boldsymbol{x}, \theta)\boldsymbol{\nabla}^2 a(\boldsymbol{x}, \theta)) + e\tilde{\boldsymbol{A}}(\boldsymbol{x}, t)\dot{\boldsymbol{x}}_i(\theta_i(t))]_{\boldsymbol{x}(\theta),\theta_i(t)}\,\mathrm{d}t \tag{4.16}$$

is a solution of the HJ equation (4.15) because of (4.11) and (4.12). We use the same c_i as (4.6) since $\boldsymbol{x}_i(t_i)$ is outside the range of the perturbation, in which case

$$\dot{\theta}_i(t_i)=1. \tag{4.17}$$

The centrifugal force is increased by a factor $\dot{\theta}^2(\boldsymbol{x}, t)$, so, to keep the trajectories as we have assumed in (4.11), the perpendicular component of the force must satisfy

$$[\boldsymbol{\nabla}\tilde{V}(\boldsymbol{x}, t)+\tilde{\boldsymbol{H}}(\boldsymbol{x}, t)\times\boldsymbol{x}'\dot{\theta}]_{\perp\boldsymbol{x}=\boldsymbol{x}_i(\theta),\theta=\theta_i(t)} = \dot{\theta}_i^2[\boldsymbol{\nabla} V(\boldsymbol{x}, \theta)+\boldsymbol{H}(\boldsymbol{x}, \theta)\times\boldsymbol{x}']_{\perp\boldsymbol{x}=\boldsymbol{x}_i(\theta),\theta=\theta_i(t)} \tag{4.18}$$

(where the magnetic fields $\boldsymbol{H}$ and $\tilde{\boldsymbol{H}}$ are respectively the curl of $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$) as a constraint on the perturbation. As a further constraint we may demand that $\nabla\theta(\boldsymbol{x}, t)$ is negligible in the parallel direction

$$\nabla\theta(\boldsymbol{x}, t) \equiv \lim_{|\boldsymbol{k}|\to 0} (\theta_j(t) - \theta_i(t))/|\boldsymbol{k}| \tag{4.19}$$

where $\boldsymbol{x}, t$ is on $\boldsymbol{x}_i(\theta_i(t))$ and $\boldsymbol{x}+\boldsymbol{k}, t$ is on $\boldsymbol{x}_j(\theta_j(t))$.

Through any point $\boldsymbol{x}$, we may plot a path in coordinate space always tangent to $\nabla S(\boldsymbol{x}, t_1) - e\boldsymbol{A}(\boldsymbol{x}, t_1)$ and this path will pass into a region where the perturbation is zero, and $\theta(\boldsymbol{x}, t)$ is just t. If $\theta(\boldsymbol{x}, t_1)$ does not change much along this curve it must remain nearly t_1 for all $\boldsymbol{x}$. Thus

$$1 - \dot{\theta}(\boldsymbol{x}, t) \ll 1. \tag{4.20}$$

We shall work to first order in $1 - \dot{\theta}(\boldsymbol{x}, t)$ and zero order in the parallel component of $\nabla\theta(\boldsymbol{x}, t)$, whenever the length L of the typical path through the region of non-zero perturbation is large:

$$L[a^2(\boldsymbol{x}, \theta)(\nabla S(\boldsymbol{x}, \theta) - e\boldsymbol{A}(\boldsymbol{x}, \theta))]_{\theta(\boldsymbol{x},t)} \cdot \nabla\theta(\boldsymbol{x}, t) \ll t(1-\dot{\theta}^2)|a^2(\boldsymbol{x}, \theta)(\nabla S(\boldsymbol{x}, \theta) - e\boldsymbol{A}(\boldsymbol{x}, \theta))|_{\theta(\boldsymbol{x},t)}. \tag{4.21}$$

To this accuracy:

$$\begin{aligned} m\,\partial a^2(\boldsymbol{x}, \theta(\boldsymbol{x}, t)) &+ \nabla \cdot [(\nabla\tilde{S}(\boldsymbol{x}, t) - e\tilde{\boldsymbol{A}}(\boldsymbol{x}, t))a^2(\boldsymbol{x}, \theta(\boldsymbol{x}, t))] \\ &= \dot{\theta}\{\partial_\theta a^2(\boldsymbol{x}, \theta)m + \nabla \cdot [(\nabla S(\boldsymbol{x}, \theta) - e\boldsymbol{A}(\boldsymbol{x}, \theta))a^2(\boldsymbol{x}, \theta)]\}_{\theta(\boldsymbol{x},t)} \\ &\quad + \partial\{[a^2(\boldsymbol{x}, \theta)(\nabla S(\boldsymbol{x}, \theta) - e\boldsymbol{A}(\boldsymbol{x}, \theta))]_{\theta(\boldsymbol{x},t)} \cdot \nabla\theta(\boldsymbol{x}, t)\} &(4.22) \\ &= 0 &(4.23) \end{aligned}$$

using the continuity equation (4.3) and (4.11);

$$\theta_i(t) = t - me\int_{t_i}^{t} [(\tilde{V}(\boldsymbol{x}, t) - V(\boldsymbol{x}, \theta))(\nabla S(\boldsymbol{x}, \theta) - e\boldsymbol{A}(\boldsymbol{x}, t))^{-2}]_{\boldsymbol{x}_i(\theta), \theta(\boldsymbol{x},t)}\, \mathrm{d}t \tag{4.24}$$

using (4.10) and (4.20); and (4.16) becomes

$$\begin{aligned} \tilde{S}(\boldsymbol{x}, t) - c_i &= P(\boldsymbol{x}, t) + \int_{t_i}^{\theta_i(t)} \dot{\theta}_i^{-1}(t)[\tfrac{1}{2}m\dot{\theta}_i^2(t)(\boldsymbol{x}')^2 - eV(\boldsymbol{x}, \theta) \\ &\quad + \tfrac{1}{2}\hbar^2 m^{-1} a^{-1}(\boldsymbol{x}, \theta)\nabla^2 a(\boldsymbol{x}, \theta) + e\dot{\theta}_i(t)\boldsymbol{A}(\boldsymbol{x}, \theta) \cdot \boldsymbol{x}']_{\boldsymbol{x}=\boldsymbol{x}_i(\theta)}\, \mathrm{d}\theta &(4.25) \\ &= P(\boldsymbol{x}, t) + \tfrac{1}{2}m\int_{t_i}^{t} (1 - \dot{\theta}_i(t))^2(\boldsymbol{x}')^2\, \mathrm{d}t \\ &\quad + \int_{t_i}^{\theta(\boldsymbol{x},t)} [\tfrac{1}{2}m(\boldsymbol{x}')^2 - eV(\boldsymbol{x}, \theta) + \tfrac{1}{2}\hbar^2 m^{-1} a^{-1}(\boldsymbol{x}, \theta)\nabla^2 a(\boldsymbol{x}, \theta) \\ &\quad + e\boldsymbol{A}(\boldsymbol{x}, \theta) \cdot \boldsymbol{x}']_{\boldsymbol{x}_i(\theta)}\, \mathrm{d}\theta + \int_{t_i}^{\theta_i(t)} (1 - \dot{\theta}_i^{-1}(t)) \\ &\quad \times [\tfrac{1}{2}m(\boldsymbol{x}')^2 + eV(\boldsymbol{x}, \theta) - \tfrac{1}{2}\hbar^2 m^{-1} a^{-1}(\boldsymbol{x}, \theta)\nabla^2 a(\boldsymbol{x}, \theta)]_{\boldsymbol{x}_i(\theta)}\, \mathrm{d}\theta &(4.26) \end{aligned}$$

where

$$P(\boldsymbol{x}, t) \equiv e \int^t m^{-1}[\tilde{\boldsymbol{A}}(\boldsymbol{x}, t) - \boldsymbol{A}(\boldsymbol{x}, \theta)] . [\nabla \tilde{S}(\boldsymbol{x}, t) - e\tilde{\boldsymbol{A}}(\boldsymbol{x}, t)]_{\boldsymbol{x}_i(\theta), \theta_i(t)} \,\mathrm{d}t$$

$$- e \int^t [\tilde{V}(\boldsymbol{x}, t) - V(\boldsymbol{x}, \theta)]_{\boldsymbol{x}_i(\theta), \theta_i(t)} \,\mathrm{d}t \qquad (4.27)$$

is a function of $\boldsymbol{x}$ as well as of t by virtue of the fact that for a given $\boldsymbol{x}$ the particular path $\boldsymbol{x}_i(\theta_i(t))$ to be used in the integral, is the one which contains the event $\boldsymbol{x}, t$.

The expression for the energy in the unperturbed system at position $\boldsymbol{x}$ and time θ is

$$\partial_\theta S(\boldsymbol{x}, \theta) = \tfrac{1}{2} m (\boldsymbol{x}')^2 + eV(\boldsymbol{x}, \theta) - \tfrac{1}{2}\hbar^2 m^{-1} a^{-1}(\boldsymbol{x}, \theta) \nabla^2 a(\boldsymbol{x}, \theta). \qquad (4.28)$$

Thus (4.26) may be written using (4.5) and (4.14) and neglecting the term of order $(1-\dot{\theta})^2$ as

$$\tilde{S}(\boldsymbol{x}, t) - \int_{t_i}^{t} \partial \tilde{S}(\boldsymbol{x}, t)\,\mathrm{d}t = S(\boldsymbol{x}, \theta(\boldsymbol{x}, t)) - \int_{t_i}^{\theta(\boldsymbol{x}, t)} [\partial_\theta S(\boldsymbol{x}, \theta)]_{\boldsymbol{x}_i(\theta)} \,\mathrm{d}\theta + P(\boldsymbol{x}, t). \qquad (4.29)$$

The HJ equation (4.15) and the continuity equation (4.23) together ensure that with $\tilde{S}(\boldsymbol{x}, t)$ defined by (4.29) and (4.14)

$$\tilde{\psi}(\boldsymbol{x}, t) = a(\boldsymbol{x}, \theta(\boldsymbol{x}, t)) \exp(\mathrm{i}\tilde{S}(\boldsymbol{x}, t)/\hbar) \qquad (4.30)$$

satisfies the Schrödinger equation

$$\mathrm{i}\hbar\, \partial \tilde{\psi}(\boldsymbol{x}, t) = [\tfrac{1}{2} m^{-1}(-\mathrm{i}\hbar\nabla - e\tilde{\boldsymbol{A}}(\boldsymbol{x}, t))^2 + e\tilde{V}(\boldsymbol{x}, t)]\tilde{\psi}(\boldsymbol{x}, t) \qquad (4.31)$$

to first order in $1 - \dot{\theta}(\boldsymbol{x}, t)$ and zero order in the parallel component $\nabla\theta(\boldsymbol{x}, t)$ given in equation (4.21).

In addition to the smoothness condition (4.18) another condition

$$\partial P(\boldsymbol{x}, t) = 0 \qquad (4.32)$$

must be satisfied (we work to zero order in the correction ∂P) if (4.29) is to be consistent with (4.14). This requires that the time dependence of $\tilde{\boldsymbol{A}}(\boldsymbol{x}, t) - \boldsymbol{A}(\boldsymbol{x}, \theta_i(t))$ and $\tilde{V}(\boldsymbol{x}, t) - \tilde{V}(\boldsymbol{x}, \theta_i(t))$ is just sufficient to compensate for the fact that different trajectories pass through $\boldsymbol{x}$ at different times when $\boldsymbol{A}$ and V are time dependent.

The time independent case is, of course, much simpler as none of the functions depend on t or on θ except trivially

$$S(\boldsymbol{x}, t) = S(\boldsymbol{x}) - Et \qquad (4.33)$$

$$\partial \tilde{S}(\boldsymbol{x}, t) = \partial_\theta S(\boldsymbol{x}, \theta) = -E \qquad (4.34)$$

$$\tilde{S}(\boldsymbol{x}, t) = \tilde{S}(\boldsymbol{x}) - Et \qquad (4.35)$$

$$\tilde{S}(\boldsymbol{x}) = S(\boldsymbol{x}) + e\int^{\boldsymbol{x}} (\tilde{\boldsymbol{A}}(\boldsymbol{x}) - \boldsymbol{A}(\boldsymbol{x})) . \mathrm{d}\boldsymbol{x} - e\int^{\boldsymbol{x}} (\tilde{V}(\boldsymbol{x}) - V(\boldsymbol{x}))\,\mathrm{d}t \qquad (4.36)$$

where the line integrals are along the trajectory $\boldsymbol{x}_i(t)$ which runs from $\boldsymbol{x}$ back into the asymptotic region. When

$$\tilde{\boldsymbol{A}}(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) = 0 \qquad (4.37)$$

equation (4.36) gives the result (2.21).

4.3. *The perturbed harmonic oscillator*

As a simple illustration of the time dependent method we take the harmonic oscillator

$$eV(\boldsymbol{x})=\tfrac{1}{2}m\omega^2\boldsymbol{x}^2 \tag{4.38}$$

as our unperturbed system,

$$\psi(\boldsymbol{x},t)=(m\omega/2\pi \mathrm{i}\hbar\sin\omega t)^{3/2}\exp\{\mathrm{i}m\omega[(\boldsymbol{x}^2+\boldsymbol{x}_0^2)\cos\omega t-2\boldsymbol{x}\cdot\boldsymbol{x}_0]/2\hbar\sin\omega t\} \tag{4.39}$$

is a solution, of the form (4.9), of the Schrödinger equation for positive t, away from $\boldsymbol{x}_0$ (Feynman and Hibbs 1965).

$$\nabla S(\boldsymbol{x},t)=m\omega(\boldsymbol{x}\cos\omega t-\boldsymbol{x}_0)/\sin\omega t \tag{4.40}$$

so the quantum paths, which in this case are also classical, are

$$\boldsymbol{x}(t)=\boldsymbol{x}_0\cos\omega t+\boldsymbol{b}\sin\omega t. \tag{4.41}$$

Along this path

$$\nabla S(\boldsymbol{x},t)|_{\boldsymbol{x}(t)}=m\omega(\boldsymbol{b}\cos\omega t-\boldsymbol{x}_0\sin\omega t) \tag{4.42}$$

$$\partial S(\boldsymbol{x},t)|_{\boldsymbol{x}(t)}=-\tfrac{1}{2}m\omega^2(\boldsymbol{b}^2+\boldsymbol{x}_0^2). \tag{4.43}$$

The solution of the Schrödinger equation for a potential $\tilde{V}(\boldsymbol{x},t)$ which differs from $V(\boldsymbol{x})$ only away from $\boldsymbol{x}_0$, which equals $\psi(\boldsymbol{x},t)$ near $\boldsymbol{x}_0$ is

$$\tilde{\psi}(\boldsymbol{x},t)=\psi(\boldsymbol{x},\theta_i(t))\exp \mathrm{i}\hbar^{-1}\left(\int_0^t \mathrm{d}\tau\,[e\tilde{V}(\boldsymbol{x}(\tau),\tau)-\tfrac{1}{2}m\omega^2\boldsymbol{x}^2(\theta_i(\tau))]\right.$$
$$\left.-\tfrac{1}{2}m\omega^2(\boldsymbol{b}^2+\boldsymbol{x}_0^2)(t-\theta_i(t))\right) \tag{4.44}$$

where

$$t-\theta_i(t)=\int_0^t (em^{-1}\omega^{-2}\tilde{V}(\boldsymbol{x}(t),t)-\tfrac{1}{2}\boldsymbol{x}^2(\theta_i(t)))(\boldsymbol{x}_0\sin\omega t-\boldsymbol{b}\cos\omega t)^{-2}\,\mathrm{d}t \tag{4.45}$$

and $\boldsymbol{x}(t)$ is given by (4.41) with $\boldsymbol{b}$ chosen so that

$$\boldsymbol{x}(t)=\boldsymbol{x}. \tag{4.46}$$

$\psi(\boldsymbol{x},t)$ is the kernel or propagator $K(\boldsymbol{x},t;\boldsymbol{x}_0,0)$ for the harmonic oscillator and $\tilde{\psi}(\boldsymbol{x},t)$ shares its singular behaviour near $\boldsymbol{x}_0$ at $t=0$, therefore

$$\tilde{K}(\boldsymbol{x},t;\boldsymbol{x}_0,0)\equiv\tilde{\psi}(\boldsymbol{x},t) \tag{4.47}$$

is the kernel for the potential $\tilde{V}(\boldsymbol{x},t)$.

In this example the exact solution of the unperturbed problem is given by the quasi-classical (WKB) approximation: neglecting the quantum potential. In general the semi-classical approximation

$$\psi(\boldsymbol{x},t)=a(\boldsymbol{x},t)\exp(\mathrm{i}S_{\mathrm{cl}}(\boldsymbol{x},t)/\hbar) \tag{4.48}$$

for the potential $V(\boldsymbol{x},t)$ is the exact quantum mechanical solution for some potential $V(\boldsymbol{x},t)+V_{\mathrm{p}}(\boldsymbol{x},t)$ where $V_{\mathrm{p}}(\boldsymbol{x},t)$ is complex and $V_{\mathrm{p}}(\boldsymbol{x})$ is the coefficient of $\psi(\boldsymbol{x},t)$ in the remainder when $\psi(\boldsymbol{x},t)$ is inserted in the Schrödinger equation for $V(\boldsymbol{x},t)$.

$V_{\mathrm{p}}(\boldsymbol{x},t)$ may be treated as a perturbation of $V(\boldsymbol{x},t)$ by the distorted wave Glauber approximation presented in §§ 3 and 4. As the method is readily extended to field theory one might start from the classical magnetic monopole solution of a unified gauge theory ('t Hooft 1974) and quantise approximately by the semi-classical (WKB) method. The result viewed as a solution of some quantum field theory may be perturbed into a solution of the unified gauge theory by the above argument.

5. Spin and quantum vector potentials

The Pauli equation

$$\mathrm{i}\hbar\,\partial\psi(\boldsymbol{x},t)=[\tfrac{1}{2}m^{-1}(-\mathrm{i}\hbar\nabla-e\boldsymbol{A}(\boldsymbol{x},t))^2+eV(\boldsymbol{x},t)-\tfrac{1}{2}e\hbar m^{-1}\boldsymbol{\sigma}\,.\,\boldsymbol{H}(\boldsymbol{x},t)]\psi(\boldsymbol{x},t) \tag{5.1}$$

may be written with

$$\psi(\boldsymbol{x},t)=a(\boldsymbol{x},t)\exp(\mathrm{i}\hbar^{-1}S(\boldsymbol{x},t))\phi(\boldsymbol{x},t) \tag{5.2}$$

where $a(\boldsymbol{x},t)$ and $S(\boldsymbol{x},t)$ are real functions and $\phi(\boldsymbol{x},t)$ is a spinor of unit magnitude

$$\phi^{\dagger}(\boldsymbol{x},t)\phi(\boldsymbol{x},t)=1 \tag{5.3}$$

as

$$\begin{aligned}&[\![[\partial S+\tfrac{1}{2}m^{-1}(\nabla S-e\boldsymbol{A})^2+eV]\phi-\tfrac{1}{2}\mathrm{i}\hbar a^{-2}\{\partial a^2+m^{-1}\nabla\,.\,[a^2(\nabla S-e\boldsymbol{A})]\}\phi\\&\quad-[\mathrm{i}\hbar\,\partial\phi+\mathrm{i}\hbar m^{-1}(\nabla S-e\boldsymbol{A})\,.\,\nabla\phi+\tfrac{1}{2}e\hbar m^{-1}\boldsymbol{\sigma}\,.\,\boldsymbol{H}\phi\\&\quad+\tfrac{1}{2}\hbar^2m^{-1}a^{-1}\nabla^2a\phi]]\!]=0.\end{aligned} \tag{5.4}$$

This becomes

$$\begin{aligned}&[\partial S+\tfrac{1}{2}m^{-1}(\nabla S-e\boldsymbol{A})^2+eV+eV_{\mathrm{q}}]\phi-\tfrac{1}{2}\mathrm{i}\hbar a^{-2}[\partial a^2+m^{-1}\nabla\,.\,a^2(\nabla S-e\boldsymbol{A}-e\boldsymbol{A}_{\mathrm{q}})]\phi\\&\quad-[\mathrm{i}\hbar\,\partial\phi+\mathrm{i}\hbar m^{-1}(\nabla S-e\boldsymbol{A})\,.\,\nabla\phi+\tfrac{1}{2}e\hbar m^{-1}\boldsymbol{\sigma}\,.\,(\boldsymbol{H}+\boldsymbol{H}_{\mathrm{q}})\phi]=0\end{aligned} \tag{5.5}$$

when we use†

$$\Phi\equiv\tfrac{1}{2}\hbar^2m^{-1}\nabla^2a\phi+\tfrac{1}{2}\mathrm{i}\hbar em^{-1}a^{-1}\phi\nabla\,.\,a^2\boldsymbol{A}_{\mathrm{q}}+eV_{\mathrm{q}}a\phi-\tfrac{1}{2}e\hbar m^{-1}a\boldsymbol{\sigma}\,.\,\boldsymbol{H}_{\mathrm{q}}\phi=0 \tag{5.6}$$

to replace $\tfrac{1}{2}\hbar^2m^{-1}a^{-1}\nabla^2a\phi$. The 'quantum potentials' are

$$e\boldsymbol{A}_{\mathrm{q}}(\boldsymbol{x},t)=\tfrac{1}{2}\mathrm{i}\hbar(\phi^{\dagger}\nabla\phi-(\nabla\phi)^{\dagger}\phi) \tag{5.7}$$

$$eV_{\mathrm{q}}(\boldsymbol{x},t)=-\tfrac{1}{4}\hbar^2m^{-1}a^{-2}(\phi^{\dagger}a\nabla^2a\phi+(\nabla^2a\phi)^{\dagger}a\phi) \tag{5.8}$$

and the 'quantum magnetic field' is defined by

$$\boldsymbol{H}_{\mathrm{q}}\,.\,\boldsymbol{\tau}=0 \tag{5.9}$$

and

$$\begin{aligned}&\phi^{\dagger}a\boldsymbol{\sigma}\nabla^2a\phi-(\nabla^2a\phi)^{\dagger}\boldsymbol{\sigma}a\phi\\&\quad=\phi^{\dagger}\boldsymbol{\sigma}\phi\nabla\,.\,a^2(\phi^{\dagger}\nabla\phi-(\nabla\phi)^{\dagger}\phi)+\phi^{\dagger}\boldsymbol{\sigma}\chi\chi^{\dagger}a\nabla^2a\phi-(\nabla^2a\phi)^{\dagger}a\chi\chi^{\dagger}\boldsymbol{\sigma}\phi\\&\quad=2e(\boldsymbol{\tau}(\nabla\,.\,a^2\boldsymbol{A}_{\mathrm{q}})-a^2\boldsymbol{H}_{\mathrm{q}}\times\boldsymbol{\tau})/\mathrm{i}\hbar\end{aligned} \tag{5.10}$$

where

$$\boldsymbol{\tau}(\boldsymbol{x},t)=\phi^{\dagger}(\boldsymbol{x},t)\boldsymbol{\sigma}\phi(\boldsymbol{x},t) \tag{5.11}$$

† With $\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_y=\begin{pmatrix}0&\mathrm{i}\\-\mathrm{i}&0\end{pmatrix}$ the proof of (5.6) is as follows. $\phi^{\dagger}a\Phi=0$ and $\mathrm{Im}\,\phi^{\dagger}a\sigma_z\Phi=0$ which imply $\Phi=D\phi$ where D is real and diagonal. $\mathrm{Im}\,\phi^{\dagger}a\sigma_x\Phi=0$ which implies D is a multiple of the identity. $\phi^{\dagger}a\Phi=0$ then implies $\Phi=0$.

and

$$\chi^{\dagger}(\boldsymbol{x}, t)\phi(\boldsymbol{x}, t) = \phi^{\dagger}(\boldsymbol{x}, t)\chi(\boldsymbol{x}, t) = 0 \tag{5.12}$$

so that ϕ and χ form a complete set in the two space at $\boldsymbol{x}, t$.

Therefore† $\chi^{\dagger}\boldsymbol{\sigma}\phi$ and $\phi^{\dagger}\boldsymbol{\sigma}\chi$ are perpendicular to $\boldsymbol{\tau}$ hence the form of the second term in (5.10). Multiplying (5.5) on the left by $\phi^{\dagger}(\boldsymbol{x}, t)$, the real and imaginary parts give, using (5.3), (5.7) and (5.9), the HJ equation

$$\partial S + \tfrac{1}{2} m^{-1}(\boldsymbol{\nabla} S - e\hat{\boldsymbol{A}})^2 + e\hat{V} = 0 \tag{5.13}$$

and the continuity equation

$$m\, \partial a^2 + \boldsymbol{\nabla} \,.\, [a^2(\boldsymbol{\nabla} S - e\hat{\boldsymbol{A}})] = 0 \tag{5.14}$$

where

$$\hat{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{A}_{\mathrm{q}} \tag{5.15}$$

$$\hat{V} = V + V_{\mathrm{q}} + V_{\mathrm{s}} - \tfrac{1}{2}\hbar m^{-1}\boldsymbol{\tau} \,.\, \boldsymbol{H} - \tfrac{1}{2} e m^{-1}\boldsymbol{A}_{\mathrm{q}} \,.\, \boldsymbol{A}_{\mathrm{q}}. \tag{5.16}$$

$$eV_{\mathrm{s}} = -\tfrac{1}{2}i\hbar(\phi^{\dagger} g\phi - (g\phi^{\dagger})\phi).$$

Defining $\boldsymbol{H}_{\mathrm{s}}$ by

$$\boldsymbol{\tau} \,.\, \boldsymbol{H}_{\mathrm{s}} = 0 \tag{5.17}$$

$$\boldsymbol{\tau} \times \boldsymbol{H}_{\mathrm{s}} = -(\boldsymbol{A}_{\mathrm{q}} \,.\, \boldsymbol{\nabla})\boldsymbol{\tau} \tag{5.18}$$

which is permissible since the change in the unit vector† $\boldsymbol{\tau}$ must be perpendicular to $\boldsymbol{\tau}$; we may write (5.5), using (5.6), (5.13) and (5.14), in the form

$$i\hbar\partial\phi + eV_{\mathrm{s}}\phi + i\hbar m^{-1}(\boldsymbol{\nabla} S - e\hat{\boldsymbol{A}}) \,.\, \boldsymbol{\nabla}\phi + \tfrac{1}{2}e\hbar m^{-1}(\boldsymbol{\sigma} - \boldsymbol{\tau}) \,.\, \hat{\boldsymbol{H}}\phi - em^{-1}(\boldsymbol{\nabla} S - e\hat{\boldsymbol{A}}) \,.\, \boldsymbol{A}_{\mathrm{q}}\phi = 0 \tag{5.19}$$

where

$$\hat{\boldsymbol{H}} = \boldsymbol{H} + \boldsymbol{H}_{\mathrm{q}} + \boldsymbol{H}_{\mathrm{s}} \tag{5.20}$$

since using an argument similar to that in the first footnote together with (5.3), (5.7) and (5.18)

$$-i\hbar e\boldsymbol{A}_{\mathrm{q}} \,.\, \boldsymbol{\nabla}\phi + \tfrac{1}{2}e\hbar\boldsymbol{\sigma} \,.\, \boldsymbol{H}_{\mathrm{s}}\phi + e^2\boldsymbol{A}_{\mathrm{q}} \,.\, \boldsymbol{A}_{\mathrm{q}}\phi = 0. \tag{5.21}$$

The Pauli equation thus implies the HJ equation (5.13), the continuity equation (5.14) and (5.19). Conversely these equations imply (5.5) and hence (5.1). The imaginary part of (5.19) multiplied on the left by $\phi^{\dagger}\boldsymbol{\sigma}$, provides its interpretation:

$$m\partial\boldsymbol{\tau} + (\boldsymbol{\nabla} S - e\hat{\boldsymbol{A}}) \,.\, \boldsymbol{\nabla}\boldsymbol{\tau} - e\boldsymbol{\tau} \times \hat{\boldsymbol{H}} = 0 \tag{5.22}$$

the equation for the change in the magnetic moment $\boldsymbol{\tau}$ of a particle travelling along the classical path for the potential $\hat{\boldsymbol{A}}$, $\hat{V}$.

The relativistic correction to the Pauli equation $-ie\hbar^2\boldsymbol{\sigma} \,.\, \boldsymbol{E} \times \boldsymbol{\nabla}\psi/4m^2$ already neglects the term proportional to $\boldsymbol{A}$, that is $-e\hbar^2\boldsymbol{\sigma} \,.\, \boldsymbol{E} \times e\boldsymbol{A}\psi/4m^2$ which appears in the Dirac equation. So *a fortiori* we may neglect all quantum corrections to $\boldsymbol{A}$ in this

† Transform to a frame in which $\phi^{\dagger}\boldsymbol{\sigma}\phi$ is in the z direction when we have chosen $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or a unimodular (cf (5.3)) multiple therefore and $\chi \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Clearly $|\boldsymbol{\tau}| \equiv |\phi^{\dagger}\boldsymbol{\sigma}\phi| = 1$ and $\phi^{\dagger}\sigma_z\chi = 0$ in this frame of reference. If $\boldsymbol{\tau}$ is a unit vector and $\phi^{\dagger}\boldsymbol{\sigma}\chi$ is perpendicular to it in one frame of reference this is true in all frames.

term. The relativistic correction then consists in replacing $\boldsymbol{H}$ in (5.20) by $\boldsymbol{H}+m^{-1}\nabla S\times\boldsymbol{E}$ or equivalently $\boldsymbol{H}+m^{-1}(\nabla S-e\hat{\boldsymbol{A}})\times\boldsymbol{E}$ to this degree of accuracy.

6. The approximation with spin

For ease of presentation the time dependent version of this section has been relegated to the appendix. We shall call the Pauli equation (5.1) with H, V and A replaced by $\tilde{H}(\boldsymbol{x})$, $\tilde{V}(\boldsymbol{x})$ and $\tilde{A}(\boldsymbol{x})$, by the name $(\widetilde{5.1})$. Its solution may be written

$$\tilde{\psi}(\boldsymbol{x})=\exp(\mathrm{i}\tilde{S}(\boldsymbol{x})/\hbar)a(\boldsymbol{x})\phi(\boldsymbol{x}) \tag{6.1}$$

satisfying $(\widetilde{5.5})$ (which is (5.5) with V, A, H, S replaced by $\tilde{V}(\boldsymbol{x})$, $\tilde{A}(\boldsymbol{x})$, $\tilde{H}(\boldsymbol{x})$, $\tilde{S}(\boldsymbol{x})$) where

$$\tilde{S}(\boldsymbol{x})=\tilde{S}_1(\boldsymbol{x})+\rho\tilde{S}_2(\boldsymbol{x}), \tag{6.2}$$

$\tilde{S}_1(\boldsymbol{x})$ and $\tilde{S}_2(\boldsymbol{x})$ are complex functions, and ρ is a constant matrix acting on $\phi(\boldsymbol{x})$, defined along with matrices $D(\boldsymbol{x})$ and $\hat{D}(\boldsymbol{x})$ as follows

$$D\phi=(D_1+\rho D_2)\phi\equiv\tfrac{1}{2}e\hbar m^{-1}(\boldsymbol{\sigma}-\boldsymbol{\tau})\,.\,\boldsymbol{H}\phi \tag{6.3}$$

$$\hat{D}\phi=(\hat{D}_1+\rho\hat{D}_2)\phi\equiv\tfrac{1}{2}e\hbar m^{-1}(\boldsymbol{\sigma}-\boldsymbol{\tau})\,.\,\hat{\boldsymbol{H}}\phi. \tag{6.4}$$

We shall proceed in a manner similar to that of §§ 2 and 4, assuming that the quantum paths for the perturbed system are the same as those for the unperturbed system but traversed at a different speed. We shall arrive at an expression for $\tilde{S}_1(\boldsymbol{x})+\rho\tilde{S}_2(\boldsymbol{x})$ which involves an integral along these paths. Each path can be divided into overlapping segments, on one of which ϕ is not an eigenstate of $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and ρ is taken to be $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. On the next segment ϕ is not an eigenstate of $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ and ρ is taken to be $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, and so on. Only one of these matrices is involved on any segment so no difficulties arise because of non-commutativity. The value of $\tilde{\psi}(\boldsymbol{x})$ and its derivative in an overlap region is taken as the boundary condition in calculating the solution in the next region. Let us suppose as we did in § 4, that

$$\nabla\tilde{S}(\boldsymbol{x})-e\hat{\tilde{\boldsymbol{A}}}(\boldsymbol{x})=\tilde{f}(x)(\nabla S(\boldsymbol{x})-e\hat{\boldsymbol{A}}(\boldsymbol{x})) \tag{6.5}$$

where

$$\hat{\tilde{\boldsymbol{A}}}=\tilde{\boldsymbol{A}}(\boldsymbol{x})+\boldsymbol{A}_{\mathrm{q}}(\boldsymbol{x}) \tag{6.6}$$

and

$$\tilde{f}(\boldsymbol{x})=\tilde{f}_1(\boldsymbol{x})+\rho\tilde{f}_2(\boldsymbol{x}) \tag{6.7}$$

with $\tilde{f}_1(\boldsymbol{x})$ and $\tilde{f}_2(\boldsymbol{x})$ being complex functions smooth along the paths tangent to $\nabla S(\boldsymbol{x})-e\hat{\boldsymbol{A}}(\boldsymbol{x})$

$$(\nabla S(\boldsymbol{x})-e\hat{\boldsymbol{A}}(\boldsymbol{x}))\,.\,\nabla\tilde{f}(\boldsymbol{x})\approx 0. \tag{6.8}$$

Then

$$\nabla\,.\,[a^2(x)(\nabla\tilde{S}(\boldsymbol{x})-e\hat{\tilde{\boldsymbol{A}}}(\boldsymbol{x}))]=0. \tag{6.9}$$

Equation $(\widetilde{5.5})$ may be written using (6.9) and (5.6) as

$$\mathrm{i}\hbar m^{-1}(\nabla\tilde{S}-e\hat{\tilde{\boldsymbol{A}}})\,.\,\nabla\phi+[-em^{-1}(\nabla\tilde{S}-e\hat{\tilde{\boldsymbol{A}}})\,.\,\boldsymbol{A}_{\mathrm{q}}(x)+\tilde{f}\hat{D}]\phi=0 \tag{6.10}$$

if $\tilde{S}(\boldsymbol{x})$ satisfies the matrix HJ equation

$$\tfrac{1}{2}m^{-1}(\nabla\tilde{S}-e\hat{\boldsymbol{A}})^2+e\hat{V}-\hat{D}+\tilde{f}\hat{D}=0 \tag{6.11}$$

with $\hat{D}$ given by (6.4) and (5.20) with $\boldsymbol{H}$ replaced by $\tilde{\boldsymbol{H}}$,

$$\hat{V}(\boldsymbol{x})\equiv\tilde{V}(\boldsymbol{x})+V_{\mathrm{q}}(\boldsymbol{x})-\tfrac{1}{2}\hbar m^{-1}\boldsymbol{\tau}(\boldsymbol{x})\cdot\tilde{\boldsymbol{H}}(\boldsymbol{x})-\tfrac{1}{2}em^{-1}\boldsymbol{A}_{\mathrm{q}}(\boldsymbol{x})\cdot\boldsymbol{A}_{\mathrm{q}}(\boldsymbol{x}). \tag{6.12}$$

If (6.11) holds, the Pauli equation $(\widetilde{5.5})$ holds since (6.10) is satisfied by virtue of (6.5) and the time-independent (5.19).

The smoothness condition, which (6.5) implies for the 'potentials' which can be read off from the HJ equation (6.11), is

$$\begin{aligned}&\nabla_{\perp}(e\hat{V}-e\hat{V}-\hat{D}+\hat{D})+em^{-1}[\nabla\times(\hat{\boldsymbol{A}}-\hat{\boldsymbol{A}})]\times(\nabla\tilde{S}-e\hat{\boldsymbol{A}})\\&\quad=(\tilde{f}^2-1)\nabla_{\perp}e\hat{V}+em^{-1}\tilde{f}(\tilde{f}-1)(\nabla\times\hat{\boldsymbol{A}})\times(\nabla S-e\hat{\boldsymbol{A}})-\nabla_{\perp}(\tilde{f}-1)\hat{D}.\end{aligned} \tag{6.13}$$

An expression for $\tilde{f}(\boldsymbol{x})$ is found from the difference between the HJ equations: (6.11) and the time independent (5.13), with $\boldsymbol{\sigma}(\boldsymbol{x})=\boldsymbol{\sigma}-\boldsymbol{\tau}(\boldsymbol{x})$

$$\begin{aligned}\tilde{f}(\boldsymbol{x})-1&=e[m^{-1}(\nabla S(\boldsymbol{x})-e\hat{\boldsymbol{A}}(\boldsymbol{x}))^2+\hat{D}(\boldsymbol{x})]^{-1}\;.\\&\quad\times[V(\boldsymbol{x})-\tilde{V}(\boldsymbol{x})-\tfrac{1}{2}\hbar m^{-1}\boldsymbol{\sigma}(\boldsymbol{x})\cdot(\boldsymbol{H}(\boldsymbol{x})-\tilde{\boldsymbol{H}}(\boldsymbol{x}))]\end{aligned} \tag{6.14}$$

to first order in $1-\tilde{f}$. The solution of the HJ equation (6.11) subject to the supposition (6.5) is, by an argument similar to that of §§ 2 and 4

$$\begin{aligned}\tilde{S}(\boldsymbol{x})&=S(\boldsymbol{x})+e\int^{\boldsymbol{x}}(\tilde{\boldsymbol{A}}(\boldsymbol{x})-\boldsymbol{A}(\boldsymbol{x}))\cdot\mathrm{d}\boldsymbol{x}-e\int^{\boldsymbol{x}}(\tilde{V}(\boldsymbol{x})-V(\boldsymbol{x}))\,\mathrm{d}t\\&\quad+\tfrac{1}{2}e\hbar m^{-1}\int^{\boldsymbol{x}}\boldsymbol{\sigma}(\boldsymbol{x})\cdot(\tilde{\boldsymbol{H}}(\boldsymbol{x})-\boldsymbol{H}(\boldsymbol{x}))\,\mathrm{d}t+\int^{\boldsymbol{x}}(1-\tilde{f}(\boldsymbol{x}))\hat{D}(\boldsymbol{x})\,\mathrm{d}t\end{aligned} \tag{6.15}$$

the line integrals being taken along that path $\boldsymbol{x}_i(t)$ which goes through $\boldsymbol{x}$.

The relativistic correction is found by replacing $\boldsymbol{H}$ by $\boldsymbol{H}+m^{-1}(\nabla S-e\hat{\boldsymbol{A}})\times\boldsymbol{E}$ in (5.16) and (5.20) and $\tilde{\boldsymbol{H}}$ by $\tilde{\boldsymbol{H}}+m^{-1}(\nabla\tilde{S}-e\hat{\boldsymbol{A}})\times\tilde{\boldsymbol{E}}$ in (5.5). This leads to an extra term $e\int^{\boldsymbol{x}}Q(\boldsymbol{x})\,\mathrm{d}t$ added to the right-hand side of (6.15) where

$$m^2Q(\boldsymbol{x})=\tfrac{1}{2}\hbar\tilde{f}(\boldsymbol{x})\boldsymbol{\sigma}(\boldsymbol{x})\cdot\nabla S(\boldsymbol{x})\times(\tilde{\boldsymbol{E}}(\boldsymbol{x})-\boldsymbol{E}(\boldsymbol{x})). \tag{6.16}$$

The (first-order) definition of $\tilde{f}(\boldsymbol{x})$ is altered by adding $Q(\boldsymbol{x})$ to $V(\boldsymbol{x})$ in (6.14) since (6.11) has an extra term eQ on the left.

It should be evident from our analysis that the method of approximation in no way supposes that the fields $\boldsymbol{H}$ and $\boldsymbol{E}$ are related to the fields $\boldsymbol{A}$ and V. This allows us to generalise away from the Pauli equation viewed as an approximation to the Dirac equation.

7. Homogeneous magnetic field

As an illustration of the distorted wave Glauber approximation with spin, we shall take as the unperturbed system, that state of a charged, $m=1$ particle in a constant homogeneous magnetic field

$$A_x=-Hy \qquad A_y=A_z=0 \tag{7.1}$$

whose Pauli equation

$$-\tfrac{1}{2}\hbar^2\left[\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\left(\frac{\partial}{\partial z}-\mathrm{i}e\hbar^{-1}Hy\right)^2+H\beta\sigma_z\right]\psi=E\psi \tag{7.2}$$

has solutions $a_n(\boldsymbol{x})\exp(\mathrm{i}\hbar^{-1}S(\boldsymbol{x}))\binom{1}{0}$ and $a_n(\boldsymbol{x})\exp(\mathrm{i}\hbar^{-1}S(\boldsymbol{x}))\binom{0}{1}$ where

$$a_n(\boldsymbol{x})=H_n[(|e|H\hbar^{-1})^{1/2}(y-y_0)]\exp[\tfrac{1}{2}|e|H(y-y_0)^2/\hbar] \tag{7.3}$$

$$S(\boldsymbol{x})=p_xx+p_zz \tag{7.4}$$

$$y_0=-p_x/eH \tag{7.5}$$

with energies

$$E_n=(n+\tfrac{1}{2})|e|\hbar H+\tfrac{1}{2}p_z^2\mp\tfrac{1}{2}\beta H\hbar^2 \tag{7.6}$$

respectively. Thus from (5.7), (5.9), (5.10), (5.17) and (5.18)

$$\boldsymbol{A}_\mathrm{q}=\boldsymbol{H}_\mathrm{q}=\boldsymbol{H}_\mathrm{s}=0. \tag{7.7}$$

$\boldsymbol{\nabla}S-e\boldsymbol{A}$ has components $(p_x-eHy, 0, p_z)$ so the path through $\boldsymbol{x}_1$ is

$$x=(p_x-eHy_1)t+x_1; \qquad y=y_1; \qquad z=p_zt+z_1 \tag{7.8}$$

in parametric form. Since $\boldsymbol{H}(\boldsymbol{x})$ has components $(0, 0, H)$ and $\boldsymbol{\tau}(\boldsymbol{x})$ components $(0, 0, \pm 1)$ it follows from the definitions (6.3) and (6.4) that

$$D(\boldsymbol{x})=\hat{D}(\boldsymbol{x})=0. \tag{7.9}$$

Thus if the *perturbation* is a scalar field $e\tilde{V}(\boldsymbol{x})$ and a (possibly unrelated) electric field $\tilde{\boldsymbol{E}}(\boldsymbol{x})$ we find using (6.15) and (6.16) that

$$\tilde{S}(x,y,z)=S(x,y,z)-e\int_0^t\tilde{V}[(p_x-eHy)t+x_1,y,p_zt+z_1]\,\mathrm{d}t+\tfrac{1}{2}e\hbar\int_0^t\boldsymbol{\sigma}^\pm\cdot\boldsymbol{p}\times\tilde{\boldsymbol{E}}[(p_x-eHy)t+x_1,y,p_zt+z_1]\tilde{f}[(p_x-eHy)t+x_1,y,p_zt+z_1]\,\mathrm{d}t \tag{7.10}$$

where t is given by (7.8) in terms of z, y, z a' d

$$\tilde{f}(x,y,z)-1=(\hbar\boldsymbol{\sigma}^\pm\cdot\boldsymbol{p}\times\boldsymbol{E}(x,y,z)-\tilde{V}(x,y,\quad))/[(p_x-eHy)^2+p_z^2], \tag{7.11}$$

σ_z^+ is $\begin{pmatrix}0 & 0\\ 0 & -2\end{pmatrix}$, σ_z^- is $\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}$, and the other components of $\boldsymbol{\sigma}^\pm$ are $\boldsymbol{\sigma}$. The smoothness condition (6.8) is satisfied if $\tilde{V}$ is smooth along the quantum path (there is a quantum potential V_q) (7.8) and (6.13) implies $\boldsymbol{\nabla}_\perp e\tilde{V}$ has components $\tilde{f}(\tilde{f}-1)(0, eHp_x, 0)$. Thus $\tilde{V}(x,y,z)$ must be small compared to $(p_x-eHy)^2+p_z^2$ at (x, y, z).

This calculation meets the case of protons scattering off nuclei or neutrons in a neutron star where the magnetic fields can be as high as 10^{12} G. The exchange force in np interactions can be included since the method can be used to develop the proton wavefunction until it enters the nuclear field and exchanges its charge. It can be used also to develop the wavefunction of the particle that has picked up the charge as it moves away from the 'stripped proton' under the influence of its nuclear field and the magnetic field. Spin–orbit coupling is introduced through an independent $\tilde{\boldsymbol{E}}$, for example

$$\tilde{\boldsymbol{E}}(\boldsymbol{x})=-\boldsymbol{\nabla}V(r)=-\frac{\boldsymbol{x}}{r}\frac{\mathrm{d}V}{\mathrm{d}r} \tag{7.12}$$

gives a spin–orbit term $\hbar^2\,\mathrm{d}V/4m^2r\,\mathrm{d}r$.

Appendix. Modifications of § 6 for time dependence

Functions here and in § 6 with a $\tilde{}$ become functions of $\boldsymbol{x}$, t; others are the functions of $(\boldsymbol{x}, t)$ in § 5 with t replaced by $\theta(x, t)$ *after* partial differentiation has been carried out. $\theta(x, t)$ is defined by (4.12) with certain restrictions on $\theta_i(t)$ to be imposed later.

The equations of § 6 are valid with the following alterations:

$$\tilde{f}\,\partial_\theta a^2 + m^{-1}\nabla \cdot a^2(\nabla\tilde{S} - e\tilde{\hat{\boldsymbol{A}}}) = 0 \tag{A6.9}$$

and

$$\mathrm{i}\hbar\tilde{f}\,\partial_\theta\phi + e\tilde{f}V_s\phi + \mathrm{i}\hbar m^{-1}(\nabla\tilde{S} - e\tilde{\hat{\boldsymbol{A}}}) \cdot \nabla\phi + [-em^{-1}(\nabla\tilde{S} - e\tilde{\hat{\boldsymbol{A}}}) \cdot \boldsymbol{A}_q + \tilde{f}\hat{D}]\phi = 0 \tag{A6.10}$$

if

$$(\nabla S - e\hat{\boldsymbol{A}}) \cdot \nabla\theta(\boldsymbol{x}, t) = 0 \tag{A.1}$$

and

$$\partial\tilde{S} + \tfrac{1}{2}m^{-1}(\nabla\tilde{S} - e\tilde{\hat{\boldsymbol{A}}})^2 + e\tilde{\hat{V}} - \tilde{\hat{D}} + \hat{D} - (1-\tilde{f})(\hat{D} + eV_s + \delta) + (1-\dot{\theta})\delta = 0 \tag{A6.11}$$

where $\tilde{V}$ is defined by (6.12) with a V_s and $\delta(\boldsymbol{x}, \theta)$ is defined by

$$\delta\phi \equiv (\delta_1 + \rho\delta_2)\phi \equiv \tfrac{1}{2}\mathrm{i}\hbar a^{-2}\phi\frac{\partial a^2}{\partial\theta} + \mathrm{i}\hbar\frac{\partial\phi}{\partial\theta}. \tag{A.2}$$

Using (6.5) and making the *ansatz*

$$\partial\tilde{S} = \partial_\theta S \tag{A.3}$$

leads to

$$\begin{aligned}\tilde{S} - \int^t (\partial\tilde{S})_{\boldsymbol{x}_i(\theta),\theta_i(t)}\,\mathrm{d}t - \tilde{f}\Big(S - \int^{\theta_i(t)} (\partial_\theta S)_{\boldsymbol{x}_i(\theta)}\,\mathrm{d}\theta\Big)\dot{\theta}^{-1} \\ = e\int^t [m^{-1}(\tilde{\boldsymbol{A}} - \boldsymbol{A}) \cdot (\nabla\tilde{S} - e\hat{A} \quad (\tilde{V} - V) \\ - \tfrac{1}{2}\hbar m^{-1}\boldsymbol{\sigma}(\boldsymbol{x}, t) \cdot (\tilde{\boldsymbol{H}} - \boldsymbol{H})]_{\boldsymbol{x}_i(\theta),\theta_i(t)}\,\mathrm{d}t \\ - \int^t [(1-\tilde{f})(\hat{D} + \delta) - (1-\dot{\theta})\delta]_{\boldsymbol{x}_i(\theta),\theta_i(t)}\,\mathrm{d}t\end{aligned} \tag{A6.15}$$

where

$$\begin{aligned}\tilde{f} = 1 - [m^{-1}(\nabla S - e\hat{\boldsymbol{A}})^2 + \hat{D} + eV_s + \delta]^{-1}[(1-\dot{\theta})\delta - e(V - \tilde{V}) \\ + e\hbar m^{-1}\boldsymbol{\sigma}(\boldsymbol{x}, t) \cdot (\boldsymbol{H} - \tilde{\boldsymbol{H}})].\end{aligned} \tag{A6.14}$$

The $\theta_i(t)$ are chosen to satisfy (A.1) and so that the right-hand side of (A6.15) is approximately time independent and so consistent with (A.3). The time dependent (6.5) demands a smoothness assumption

$$\begin{aligned}\nabla_\perp[e\tilde{\hat{V}} - e\hat{V} - \tilde{\hat{D}} + \hat{D} + (1-\dot{\theta})\delta] + m^{-1}e[\nabla \times (\tilde{\hat{\boldsymbol{A}}} - \hat{\boldsymbol{A}})] \times (\nabla\tilde{S} - e\tilde{\hat{\boldsymbol{A}}}) \\ = (\tilde{f}^2 - 1)\nabla_\perp e\hat{V} + em^{-1}\tilde{f}(\tilde{f} - 1)(\nabla \times \hat{\boldsymbol{A}}) \times (\nabla S - e\hat{\boldsymbol{A}}) \\ - \nabla_\perp(\tilde{f} - 1)(\hat{D} + eV_s + \delta).\end{aligned} \tag{A6.13}$$

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